MEC 455 – APPLIED STRESS ANALYSIS T. Nakamura

TWO-DIMENSIONAL (PLANE) ELASTICITY

- Equilibrium Equations (without body force)

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0 \qquad \left(\text{with body force} \qquad \frac{\partial \sigma_{ij}}{\partial x_i} + B_i = 0\right)$$

$$\Longrightarrow \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \ , \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

- Strain–Displacement Relation

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\Longrightarrow \quad \epsilon_x = \frac{\partial u_x}{\partial x} \ , \qquad \epsilon_y = \frac{\partial u_y}{\partial y} \ , \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Compatibility of Strain

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

- Stress–Strain Relation

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

Plane Stress $[\sigma_z = \tau_{zx} = \tau_{zy} = 0]$

$$\epsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\epsilon_y = \frac{1+\nu}{E}\sigma_y - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

Plane Strain $[\sigma_z = \nu(\sigma_x + \sigma_y),\ \tau_{zx} = \tau_{zy} = 0]$

$$\epsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu(1+\nu)}{E}(\sigma_x + \sigma_y)$$

$$\epsilon_y = \frac{1+\nu}{E}\sigma_y - \frac{\nu(1+\nu)}{E}(\sigma_x + \sigma_y)$$

$$\epsilon_z = 0$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

Airy Stress Function $[\Phi = \Phi(x,y) = \Phi(r,\theta)]$

Cartesian Coordinates

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} \ , \qquad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} \ , \qquad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

$$\bigtriangledown^2(\bigtriangledown^2 \Phi) = \frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial^2 x \partial^2 y} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Polar Coordinates

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \ , \qquad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} \ , \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)$$

$$\bigtriangledown^2(\bigtriangledown^2 \Phi) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Phi = 0$$