Structured H2 Optimization of Vehicle Suspensions Based on Multi-Wheel Models

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SUMMARY

Various control techniques, especially LQG optimal control, have been applied to the design of active and semi-active vehicle suspensions over the past several decades. However, passive suspensions remain dominant in the automotive marketplace because they are simple, reliable, and inexpensive. The force generated by a passive suspension at a given wheel can depend only on the relative displacement and velocity at that wheel, and the suspension parameters for the left and right wheels are usually required to be equal. Therefore, a passive vehicle suspension can be viewed as a decentralized feedback controller with constraints to guarantee suspension symmetry.

In this paper, we cast the optimization of passive vehicle suspensions as structure-constrained LQG/H2 optimal control problems. Correlated road random excitations are taken as the disturbance inputs; ride comfort, road handling, suspension travel, and vehicle-body attitude are included in the cost outputs. We derive a set of necessary conditions for optimality and then develop a gradient-based method to efficiently solve the structure-constrained H2 optimization problem. An eight-DOF four-wheel-vehicle model is studied as an example to illustrate application of the procedure, which is useful for design of both passive suspensions and active suspensions with controller-structure constraints.

1. INTRODUCTION

In the past three decades, a great deal of research has been carried out into the optimization of vehicle suspensions. A classified bibliography, including nearly 600 papers, is presented by Elbeheiry et al. [1]. Comprehensive surveys can also be found in the papers by Hedrick and Wormely [2], Sharp and Crolla [3], Karnopp [1], and Hrovat [5].

A vehicle suspension is required to give good ride comfort and road handling accompanied by small motion of the vehicle body in a limited working space. Excitation of vehicle vibration is primarily due to road irregularities, which are well characterized as random; therefore, stochastic optimal control can be used to make a

With the development of electronics and microprocessors, commercial automobiles with active suspensions became available in the 1990s. Although active suspensions can improve the ride comfort and road handling beyond that attainable by passive suspensions, the cost, weight, and power requirements of active suspensions remain prohibitive. Because passive suspensions are simple, reliable, and inexpensive, they remain dominant in the automotive marketplace.

Compared to the tremendous volume of research published on the optimization of active suspensions, there have appeared relatively few studies on optimization of passive suspensions. Li and Pin [20] employ evolutionary algorithms to optimize a passive quarter-car suspension. Optimization of a quarter-car suspension is formulated as an H2 optimal control problem by Corriga et al. [21] and a simplex direct search is employed to find the optimum values of two parameters. Camino et al. [22] apply a linear-matrix-inequality (LMI) based min/max algorithm for static output feedback to the design of passive H2 optimal quarter-car suspensions.

Whereas the control forces generated at each wheel of an active suspension can be based on all of the sensor signals employed in the system, the forces generated at a given wheel of a passive suspension can depend only on the relative displacement and velocity at that wheel. This “decentralized” architecture of passive systems makes optimization of passive suspensions based on multi-wheel models considerably more difficult than optimization of the corresponding active suspension. Elbeheiry et al. [18] obtain suboptimal designs of both passive and active suspensions based on full-car models. By
minimizing the variance of the control force difference between the passive suspension and the LQG active suspension with full-state feedback, Lin and Zhang [23] obtain the suboptimal parameters of LQG passive suspensions based on a half-car model.

General nonlinear programming approaches (especially direct search methods) are relatively inefficient, but can be used to optimize systems with controller-structure and parameter-value constraints as well as non-linear dynamics. Elmadany [24] develops a procedure based on covariance analysis and a direct search method to optimize the passive suspension of a three-axle half-vehicle model. Castillo et al. [25] use sequential linear programming to minimize the weighted acceleration of the passenger subject to a constraint on the suspension stroke. Non-linear characteristics of dampers have been taken into account by Demic [26] using the modified Hooke-Jeeves method and by Spentzas [27] using Box’s method.

In this paper, we use decentralized LQG/H2 optimization to design passive and structure-constrained active suspensions based on general linear or linearized vehicle models. We take the ride comfort, road handling, suspension deformation, and vehicle-body attitude as performance indices; trade-offs among the various performance requirements are made using weighting factors. The problem of passive-suspension design is formulated as a decentralized feedback design problem by casting the springs and dampers, respectively, as local position and velocity feedback elements. Introduction of a Lagrange multiplier matrix enables derivation of a set of necessary conditions for optimality and efficient computation of the gradient of the cost with respect to the free design variables. This forms the basis for direct and efficient optimization of passive suspensions based on general (quarter-car, half-car, or full-multi-axle) vehicle models. This method is also useful for active suspensions with decentralized feedback, state-limited feedback, or other constraints on the controller architecture. As an example, we use the method to design both a passive suspension and an active static output feedback suspension based on a full-car (four-wheel) model.

2. PROBLEM FORMULATION

In this section, we show that design of a passive suspension is equivalent to design of a controller with structured static output feedback. The feedback gains are composed of the stiffness and damping parameters associated with each wheel, the closed-loop inputs are the ground disturbances, and the closed-loop outputs are the performance indices.

2.1. Passive Suspensions and Decentralized Feedback

The shock absorbers are approximated as linear dashpots, ignoring the asymmetry in the jounce and rebound. The tire is modeled as a linear spring with or without some
small damping. The wheel is taken as a one-DOF mass, and the vehicle body is modeled as a rigid body. Figure 1 shows a typical vehicle model with eight DOF. Design of a passive mechanical system composed of masses, springs, and dampers can be cast as a decentralized control problem [28] by recognizing that springs and dampers, respectively, feed back the relative displacements and velocities locally. Replacing the forces generated by the passive suspension elements with a control force vector \( u \), we can write the equations governing vibration of the vehicle as

\[
\begin{align*}
M_\omega \ddot{q} + C_\omega \dot{q} + K_\omega q &= B_u u + B_p q_0 + B_v \dot{q}_0 \\
\end{align*}
\]

where \( M_\omega, C_\omega, \) and \( K_\omega \) are the mass, damping, and stiffness matrices, respectively; \( q_0 \) and \( \dot{q}_0 \) are the disturbance vectors arising from the ground displacement and velocity; \( B_u, B_p, \) and \( B_v \) are constant matrices with appropriate dimensions.

Defining the state variables as

\[
\dot{x} = \begin{bmatrix}
q \\
\dot{q} - M_\omega^{-1} B_v q_0
\end{bmatrix}
\]

we can write the equation of motion (1) as

\[
\dot{\dot{x}} = \begin{bmatrix}
0 & I \\
-M_\omega^{-1} K_\omega & -M_\omega^{-1} C_\omega
\end{bmatrix} \dot{x} + \begin{bmatrix}
M_\omega^{-1} B_v \\
M_\omega^{-1} (B_p - C_\omega M_\omega^{-1} B_v)
\end{bmatrix} q_0 + \begin{bmatrix}
0 \\
M_\omega^{-1} B_u
\end{bmatrix} u
\]

\[
\text{def } \hat{A} \dot{x} + \hat{B}_1 q_0 + \hat{B}_2 u
\]

Fig. 1. A typical four-wheel vehicle model.
Based on the geometry of the vehicle, we write the vector of “measured” outputs—the relative displacements and velocities at the suspension connections—as a linear combination of the states and inputs; that is,

\[ y = \bar{C}_2 \bar{x} + \bar{D}_{21}q_0 + \bar{D}_{22}u \]  

(4)

in which the matrix \( \bar{D}_{22} \) turns out to be zero naturally in passive suspension design. Similarly, we can write the vertical velocities of the driver and vehicle body, suspension deformation, and dynamic contact force as an output vector in the form

\[ \bar{z} = \bar{C}_1 \bar{x} + \bar{D}_{11}q_0 + \bar{D}_{12}u \]  

(5)

The forces generated by the suspension springs and dampers are determined from \( y \) according to

\[ u = F_d y \]  

(6)

where the “feedback gain” \( F_d \) is a decentralized (block-diagonal) matrix composed of the suspension parameters to be optimized. For example, for a four-wheel vehicle model, the feedback gain takes the form

\[
F_d = \begin{bmatrix}
  k_{fl} & c_{fl} & & \\
  & k_{fr} & c_{fr} & \\
  & & k_{rl} & c_{rl} \\
  & & & k_{rr}
\end{bmatrix}
\]  

(7)

where \( k_{fl} \) and \( c_{fl} \) denote, respectively, the stiffness and damping of the suspension at the front-left wheel, \( k_{fr} \) and \( c_{fr} \) denote those at the front-right wheel, and so on. To preserve vehicle symmetry, we constrain the suspension parameters corresponding to the left and right sides of the vehicle to be equal:

\[
k_{fl} = k_{fr}, \quad k_{rl} = k_{rr}, \quad c_{fl} = c_{fr}, \quad \text{and} \quad c_{rl} = c_{rr}
\]  

(8)

### 2.2. Road-Roughness Excitation

The disturbances acting on the vehicle suspension system include road irregularities, braking forces, acceleration forces, inertial forces on a curved track, and payload changes. Among them, road roughness is the most important disturbance to either the rider or the vehicle structure itself [1]. Many road surface profiles have been measured, and several road models have been discussed in the literature [3, 5]. In the context of vibration, the road roughness is typically represented as a stationary Gaussian stochastic process of a given displacement power spectral density (PSD) in m²/(cycle/m):

\[ S_{psd}(\nu) = G_r \nu^\beta \]  

(9)
where \( \nu \) is the spatial frequency, \( G_r \) is the road-roughness coefficient, and the exponent \( \beta \) is commonly approximated as \(-2\). The International Organization for Standardization (ISO) suggests a road classification scheme based on the value of \( G_r \), as shown in Table 1 [11, 18]. When a vehicle is driven at a constant speed \( V \), the temporal excitation frequency \( \omega \) and the spatial excitation frequency \( \nu \) are related by \( \omega = 2\pi \nu \). And the displacement power spectral density in terms of temporal frequency can be obtained by using \( S(\omega) d\omega = S(\nu) d\nu \) [3]:

\[
S_{psd}(\omega) = \frac{2\pi G_r V}{\omega^2}
\]

But the road-roughness characteristic given by (9) is not valid at very low spatial frequencies. Thus a cutoff \( \nu_0 \) between 0.001 and 0.02 cycle/m is used to limit the displacement to be finite at vanishingly small spectral frequencies, and we modify (10) to become

\[
S_{psd}(\omega) = \frac{2\pi G_r V}{\omega^2 + \omega_0^2}
\]

where \( \omega_0 = 2\pi \nu_0 \). This displacement disturbance to the vehicle tire can be represented by a white noise signal \( w(t) \) passing through a first-order filter given by

\[
G(s) = \frac{(2\pi G_r V)^{1/2}}{s + \omega_0}
\]

For a full- or half-car model, the vehicle will be excited by the road irregularities at more than one location. The excitations at the left and right wheels are correlated at low frequencies and uncorrelated at high frequencies. A two-dimensional road-roughness model is proposed by Rill [29] and has been used by Crolla and Abdel-Hady [15] and Elbeheiry et al. [18] in active suspension design. Crolla and Abdel-Hady [15] find that the correlation of excitations acting on the left and right wheels is not important for design; therefore in the following we ignore it. For most multi-axle vehicles, the distances between the wheels at different axles differ little. Hence it is reasonable to take the excitations at the rear axle as a pure delay of those at the front axle, with a delay time \( t_0 = L/V \), where \( L \) is the distance between axles.
2.3. Performance Indices

The suspension plays an important role in determining many aspects of the performance of a vehicle, especially the ride comfort, road handing, and the vehicle attitude. Ride comfort is measured by a specific index, which depends on the acceleration level, frequency, direction, and location. The ISO 2631 standard [30] specifies a method of evaluation of the effect exposure to vibration on humans by weighting the root-mean square (RMS) acceleration with human vibration-sensitivity curves. The frequency weighting curve for vertical acceleration (measured at the seat surface) is shown in Figure 2. A second-order shape filter of the form

\[ H_{2631}(s) = \frac{50s + 500}{s^2 + 50s + 1200} \]

has been used in [18, 24] to approximate this ISO weighting curve and is plotted in Figure 2. Although it is possible to design other filters to better approximate the ISO weighting curve [31], we adopt this filter for the purposes of this paper.

One measure of road handing is obtained from the dynamic contact forces between the tires and the ground, because maintenance of large contact forces is necessary to maintain the traction required for braking, acceleration, and steering. The dynamic contact forces can be calculated from the deformation and stiffness of each tire. For a

![Fig. 2. Human vibration sensitivity weighting curve for vertical acceleration: ISO2631-1 (solid), approximate second-order filter (dotted).](image-url)
full-vehicle model, the attitude includes the velocities of heave, pitch, and roll of the vehicle body. The working space of a suspension is bounded by stops, so a small suspension deformation is preferred. The vector of cost outputs should include all of the performance requirements: the weighted acceleration of the passenger, vehicle-body attitude, dynamic contact force of the tires, and suspension deformation.

2.4. Problem Statement

We combine the vehicle model, road model, and performance requirements discussed in the foregoing to formulate a decentralized control problem for design of passive suspensions as shown in Figure 3. Using Padé expansions to approximate the delay, we obtain a generalized LTI plant model including all of the shape filters and weighting factors:

\[
\dot{x} = Ax + B_1w + B_2u, \quad z = C_1x + D_{12}u, \quad y = C_2x
\]  

where the ground inputs are replaced by a white-noise input \( w \), the state vector \( x \) is obtained by augmentation of \( \bar{x} \) from Equation (2) to include the states necessary for incorporation of the shape filters, and likewise, the coefficient matrices \( A, B_1, B_2, C_1, D_{12}, \) and \( C_2 \) are obtained by augmentation of the corresponding (barred) matrices in (3)–(5). Our task is to design the decentralized controller law \( u = F_dy \) in order to minimize the H2 norm of the closed-loop system from \( w \) to \( z \), where \( F_d \) is of the block

![Fig. 3. Structured control formulation of a passive suspension system.](image)
diagonal form given by Equation (7) subject to the symmetry constraints given by Equation (8). (In the following section, we show that H2 optimization is equivalent to minimization of the RMS response to a random road excitation.) The “closed-loop” system from $w$ to $z$ can be written in the standard form

$$
\dot{x} = A_c x + B_c w, \quad z = C_c x + D_c w
$$

where

$$
\begin{bmatrix}
A_c & B_c \\
C_c & D_c
\end{bmatrix} = 
\begin{bmatrix}
A + B_2 F_d C_2 & B_1 \\
C_1 + D_1 F_d C_2 & D_1 F_d C_2
\end{bmatrix}
$$

3. STRUCTURED H2 OPTIMIZATION

In this section, we introduce the system H2 norm and then present a gradient-based approach for H2 optimization with decentralized feedback and symmetric structure constraints.

3.1. System H2 Norm

For a LTI system with a state-space realization such as that of Equation (15), the H2 norm is defined as the signal 2-norm of the system impulse response matrix $h_{zw}$:

$$
\|H\|_2^2 = \int_0^\infty \text{trace}[(h'_{zw}(t)h_{zw}(t))] dt
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[(H'_{zw}(j\omega)H_{zw}(j\omega))] d\omega
$$

where $H_{zw}$ is the system transfer matrix and the primes denote conjugate matrix transposition. If the system input $w(t)$ is unit white noise characterized by $E[w(t)] = 0$ and $E[w(t)w(t-\tau)] = \delta(\tau)$ where $\delta$ is the Dirac delta, we obtain from Equation (17) that

$$
\lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T z(t)'z(t) \, dt \right] = \|H\|_2^2
$$

from which we see that the system H2 norm is the asymptotic value of the output variance (i.e., the RMS value of the system output) with unit white noise input. From Equation (14), we have $z'(t)z(t) = x' C_1' C_1 x + 2 x' C_1' D_1 u + u' D_1' D_1 u$; thus, H2 and LQG optimization are formally equivalent. Moreover, because the system H2 norm is a measure of the RMS response to random excitation, it is an appropriate measure of the performance of an automotive suspension.
The impulse response of the closed-loop LTI system given by Equation (15) is
\[ h_{zw}(t) = C_c e^{A_c t} B_c + D_c \delta(t) \]. Substitution of this expression into (17) yields the following theorem [32]:

**Theorem 1** The H2 norm from \( w \to z \) of the LTI system given by Equation (15) is
finite if \( A_c \) is unstable or \( D_c \neq 0 \), otherwise

\[ \|H\|_2^2 = \text{trace}(C_c PC_c^t) = \text{trace}(B_c^t KB_c) \]  
(19)

where \( P \) and \( K \) are, respectively, the controllability and observability Grammians, which satisfy the Lyapunov equations

\[ A_c P + PA_c^t + B_c B_c^t = 0 \]  
(20)

\[ A_c^t K + KA_c + C_c^t C_c = 0 \]  
(21)

Theorem 1 provides an approach to evaluate the system H2 norm (equivalent to the RMS response of the system to unit white noise input) and provides a starting point for H2-optimal controller synthesis. For full-state feedback, the optimal gains can be obtained by solving an algebraic Riccati equation. If only some outputs are measured, an optimal full-order controller can be obtained by solving two decoupled Riccati equations for the feedback gain and the Kalman state estimator [32]. However, with controller-structure constraints, such direct solutions are not available.

### 3.2. Structured H2 Optimization

Passive suspension design is equivalent to design of a controller with a decentralized architecture and additional constraints on the symmetry of the vehicle and the ranges of the design parameters. Structure-constrained (usually referred to as “structured”) control optimization has been studied by a number of researchers in the control community. Levene and Athans [33] obtain necessary conditions for optimality of (centralized) static output-feedback controllers (i.e., controllers that feed back a linear combination of outputs). Mendel [34] employs Lagrange matrix multipliers to derive the necessary conditions for standard LQG problems with centralized static output feedback. The optimal static output H2/LQG gains can be computed using nonlinear programming [35] or linear matrix inequality (LMI) iteration [36]. The decentralized H2 optimal control gains can be computed using gradient-projection methods [37, 38], multiplier methods [39], and penalty function methods [40]. In the following, we adapt the method of Mendel [34] to develop an efficient and direct method for optimization of controllers with more general constraints.
Based on Theorem 1, the H2 norm of the closed-loop system is given by 
\[ \text{trace}(B'_1 KB_1) \]
where \( K \) is a symmetric matrix satisfying
\[
K(A + B_2 F_d C_2) + (A + B_2 F_d C_2)' K
+ (C_1 + D_{12} F_d C_2)' (C_1 + D_{12} F_d C_2) = 0 \tag{22}
\]
Therefore, static output H2 control with controller architecture constraints becomes a constrained optimization problem:
\[
\min J(F_d) = \|H_{ov}\|^2_2 = \text{trace}(B'_1 KB_1) \tag{23}
\]
s.t. (22) and \( F_d \in S_f \)
where \( S_f \) is the set of matrices which have the prescribed architecture and stabilize the closed-loop system.

We now define a Lagrangian function of the form
\[
\mathcal{L}(F_d, K, L) = \text{trace}\{ (B'_1 KB_1) \\
+ [K(A + B_2 F_d C_2) + (A + B_2 F_d C_2)' K \]
\[
+ (C_1 + D_{12} F_d C_2)' (C_1 + D_{12} F_d C_2)] L \}
\]
where \( L \) is a (symmetric) Lagrange multiplier matrix. Then, using matrix calculus [41], we write
\[
\frac{\partial \mathcal{L}}{\partial F_d} = 2(D_{12}' D_{12} F_d C_2 + D_{12}' C_1 + B'_2 K) LC_2' \tag{25}
\]
\[
\frac{\partial \mathcal{L}}{\partial L} = K(A + B_2 F_d C_2) + (A + B_2 F_d C_2)' K
+ (C_1 + D_{12} F_d C_2)' (C_1 + D_{12} F_d C_2) \tag{26}
\]
\[
\frac{\partial \mathcal{L}}{\partial K} = L(A + B_2 F_d C_2)' + (A + B_2 F_d C_2) L + B_1 B'_1 \tag{27}
\]
The gradient \( \partial \mathcal{L} / \partial F_d \) is an ensemble of the \( \partial \mathcal{L} / \partial F_{d_{ij}} \), so the right-hand side of Equation (25) is not really the derivative of \( \mathcal{L} \) with respect to the free design variables in \( F_d \). We must also take account of the symmetry constraint Equation (8). Introducing the notation \( k_{il} = k_{rl} := k_r, c_{il} = c_{rl} := c_r \) and \( c_{il} = c_{rr} := c_r \), we use the chain rule to write
\[
\frac{\partial \mathcal{L}}{\partial k_f} = \frac{\partial \mathcal{L}}{\partial k_{fl}} + \frac{\partial \mathcal{L}}{\partial k_{fr}} = (\frac{\partial \mathcal{L}}{\partial F_d})_{1,1} + (\frac{\partial \mathcal{L}}{\partial F_d})_{2,3} \tag{28}
\]
\[
\frac{\partial \mathcal{L}}{\partial k_r} = \frac{\partial \mathcal{L}}{\partial k_{rl}} + \frac{\partial \mathcal{L}}{\partial k_{rr}} = (\frac{\partial \mathcal{L}}{\partial F_d})_{3,5} + (\frac{\partial \mathcal{L}}{\partial F_d})_{4,7} \tag{29}
\]
\[
\frac{\partial \mathcal{L}}{\partial c_f} = \frac{\partial \mathcal{L}}{\partial c_{fl}} + \frac{\partial \mathcal{L}}{\partial c_{fr}} = (\frac{\partial \mathcal{L}}{\partial F_d})_{1,2} + (\frac{\partial \mathcal{L}}{\partial F_d})_{2,4} \tag{30}
\]
\[
\frac{\partial \mathcal{L}}{\partial c_r} = \frac{\partial \mathcal{L}}{\partial c_{rl}} + \frac{\partial \mathcal{L}}{\partial c_{rr}} = (\frac{\partial \mathcal{L}}{\partial F_d})_{2,4} + (\frac{\partial \mathcal{L}}{\partial F_d})_{4,8} \tag{31}
\]
where \((\partial L/\partial F_{d})_{ij}\) denotes the \(ij\)th entry of \(\partial L/\partial F_{d}\) in Equation (25). Similar expressions can be derived for other multi-wheel vehicle models, such as those for three-axle vehicles.

Thus a set of necessary conditions for H2 optimization of passive vehicle suspensions can be written as

\[
\frac{\partial L}{\partial L} = 0, \quad \frac{\partial L}{\partial K} = 0
\]

(32)

\[
\frac{\partial L}{\partial k_f} = 0, \quad \frac{\partial L}{\partial k_r} = 0, \quad \frac{\partial L}{\partial c_f} = 0, \quad \frac{\partial L}{\partial c_r} = 0
\]

(33)

\[A + B_2F_dC_2\text{ is stable}\]

It is not trivial to solve this set of highly nonlinear equations, but in the following we develop a gradient-based solution method.

For a given \(F_d\) (comprising \(k_f, k_r, c_f,\) and \(c_r\)), \(\partial L/\partial L = 0\) and \(\partial L/\partial K = 0\) are two decoupled Lyapunov equations, which can be solved easily. Further, if these equations are satisfied, we have

\[
\frac{\partial J}{\partial k_f} = \frac{\partial L}{\partial k_f}, \quad \frac{\partial J}{\partial k_r} = \frac{\partial L}{\partial k_r}, \quad \frac{\partial J}{\partial c_f} = \frac{\partial L}{\partial c_f} \quad \text{and} \quad \frac{\partial J}{\partial c_r} = \frac{\partial L}{\partial c_r}
\]

(35)

That is, after solving the two Lyapunov equations for a given \(F_d\), we obtain the gradient of \(J = \|H_{zw}\|_2^2\) with respect to the design variables using Equations (28)–(31). Therefore, starting with a stabilizing \(F_d\), we can use a gradient-based optimization method, such as steepest-descent, conjugate-gradient, or FBGS quasi-Newton methods, to solve for a (locally) optimal H2 feedback gain. For details about gradient-based optimization, please refer to the text by Bertsekas [42].

For passive suspension systems, the stiffness and damping coefficients are nonnegative. To handle this additional constraint that \(F_{dij} \geq 0\), we can conveniently replace the \(F_{dij}\) with \(F_{dij}^2\), and make a corresponding modification to the gradient. More generally, if we would like to constrain some parameter \(F_{dij}\) to be in some reasonable interval \([r_1, r_2]\), we can specify \(F_{dij}\) with one parameter \(r\) as in

\[
F_{dij} = \frac{1}{2}(r_1 + r_2) + \frac{1}{2}(r_2 - r_1)\sin(\alpha r)
\]

(36)

where \(\alpha\) is a scaling coefficient chosen to accelerate convergence, and the gradient is evaluated using the chain rule.

4. DESIGN EXAMPLE: FULL-CAR MODEL

To illustrate the decentralized H2 optimization for passive suspensions, in the following we take an eight-DOF full-car model as an example. Such a model has been
used by Kim and Yoon [17] for a semi-active suspension system. The parameters are shown in Table 2, in which the origin of the coordinate system is chosen to be the center of mass of the sprung mass (vehicle body), as shown in Figure 1. We use a 4th-order Pade expansion to approximate the delay, and the second-order filter given by Equation (13) to approximate the ISO 2631 frequency weighting curve. The spatial cut-off frequency $\nu_0$ is taken to be 0.005 cycle/m. We obtain a 28th order generalized plant, in which the disturbance input $w(t)$ is a white-noise vector $[w_1(t), w_2(t)]'$, the “measured” outputs for feedback are

$$y(t) = [d_f, \dot{d}_f, d_r, \dot{d}_r, d_l, \dot{d}_l, d_{rl}, \dot{d}_{rl}]'$$ (37)

and the cost outputs include each of the weighted performance indices:

$$z(t) = [e_1 \ddot{z}_p, e_2 \dot{z}_b, e_3 \dot{\theta}_x, e_4 \dot{\theta}_y, e_5 d_f, e_6 \dot{d}_f, e_7 d_r, e_8 \dot{d}_r, e_9 f_f, e_{10} f_r, e_{11} f_l, e_{12} f_{rl}]'$$ (38)

where $\ddot{z}_p$ represents the acceleration $\ddot{z}_p$ at the driver’s seat passed through the ISO 2631 filter, and the $e_i$ are weighting factors. Here, we choose them to be

<table>
<thead>
<tr>
<th>Weighting Factor</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
<th>$e_5$–$e_8$</th>
<th>$e_9$–$e_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>1</td>
<td>8.3</td>
<td>8.3</td>
<td>120</td>
<td>120</td>
<td>8.3e-3</td>
</tr>
</tbody>
</table>
4.1. Design of a Passive Suspension

Using the proposed method for structured H2 optimization and setting as the stop criterion that each of the gradients $\partial J/\partial k_f$, $\partial J/\partial k_r$, $\partial J/\partial c_f$, and $\partial J/\partial c_r$ be less than $10^{-4}$, we obtain the optimal parameters for the passive suspension at various vehicle speeds, as plotted in Figure 4. Figure 5 shows the minimal closed-loop H2 norm achieved, that is, the total weighted RMS values of the vehicle driven at various speeds on a class B road. The vehicle attitudes and the RMS acceleration of the driver seat weighted by the ISO 2631 filter are shown in Figure 6. The suspension deformations and dynamic tire-ground contact forces are shown in Figure 7. For a purely passive (not adaptive) design, we must set the suspension parameters to constant values independent of the vehicle speed. We fix the passive suspension

![Graph of optimal parameters vs. vehicle speed](image)

Fig. 4. Optimal parameters of the passive suspension as a function of vehicle speed: front (solid line), rear (dashed).
parameters to those optimized for 30 m/s:

\[
F_d = \begin{bmatrix}
17358 & 2380 \\
17358 & 2380 \\
19181 & 2429 \\
19181 & 2429 \\
\end{bmatrix}
\]  

(39)

and plot the achieved performances over the range of speeds in Figures 5–7. As a comparison, Figures 5–7 also show the performances achieved with the nominal parameters given in Table 2 and obtained from reference [17]. As expected, the passive suspensions obtained from the H2 optimization yield better overall performance than the nominal design. For a fixed-parameter suspension, the weighted acceleration of the driver’s seat, dynamic tire-ground contact forces, and suspension deformations will increase monotonically as the vehicle speed increases, but the pitch velocity of the vehicle reaches a maximum at an intermediate speed due to the delay in the disturbances. It is interesting to note that the fixed-parameter passive suspension optimized for a properly selected speed yields ride comfort and road handling close to those of the optimal (adaptive) suspension over a large range of speeds.
4.2. Active Suspensions with Static Output Feedback

For comparison, we also design optimal H2 active suspensions using static feedback of the output \( y(t) \) as in Equation (37) (suspension deformations and relative velocities). For widely selected initial guesses, the gradient-based approach yields the following optimal output feedback gain at a vehicle speed of 30 m/s:

\[
F = \begin{bmatrix}
15260 & 2121 & -1733 & -187 & 963 & 155 & -10492 & 347 \\
-2142 & -166 & 15406 & 2102 & -10951 & -339 & -766 & 154 \\
-15759 & 1893 & 1839 & 44 & -13045 & -288 & 39054 & 2025
\end{bmatrix}
\]

The achieved performances on a class B road with the optimal active suspension, optimal passive suspension, and nominal passive suspension are compared in Table 3.
Fig. 7. Suspension deformation and dynamic tire-ground contact forces on a class B road: optimal (solid), nominal (dotted), optimized for 30 m/s (dash-dot). The front and rear suspension deformations and contact forces shown here are the averages of the left and right values.

Table 3. Performances achieved with active and passive suspensions at $V = 30 \text{ m/s}$ on a class B road.

<table>
<thead>
<tr>
<th>Performance</th>
<th>Optimal active</th>
<th>Optimal passive</th>
<th>Nominal passive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total H2 norm</td>
<td>5.853</td>
<td>6.652</td>
<td>7.142</td>
</tr>
<tr>
<td>RMS acceleration $\ddot{x}_p$, m/s$^2$</td>
<td>0.1976</td>
<td>0.2275</td>
<td>0.2296</td>
</tr>
<tr>
<td>RMS suspension travel, mm</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Front ave.</td>
<td>4.781</td>
<td>5.083</td>
<td>6.542</td>
</tr>
<tr>
<td>Rear ave.</td>
<td>3.308</td>
<td>3.837</td>
<td>5.186</td>
</tr>
<tr>
<td>RMS normalized dynamic contact force, %</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Front ave.</td>
<td>8.764</td>
<td>8.827</td>
<td>9.565</td>
</tr>
<tr>
<td>Rear ave.</td>
<td>8.708</td>
<td>11.342</td>
<td>12.075</td>
</tr>
<tr>
<td>RMS vehicle body motion</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heave, cm/s</td>
<td>2.282</td>
<td>3.078</td>
<td>3.693</td>
</tr>
<tr>
<td>Pitch, $\times 10^{-2} \text{ rad/s}$</td>
<td>0.450</td>
<td>1.002</td>
<td>0.789</td>
</tr>
<tr>
<td>Roll, $\times 10^{-2} \text{ rad/s}$</td>
<td>3.683</td>
<td>4.468</td>
<td>5.195</td>
</tr>
</tbody>
</table>
The frequency response of the weighted acceleration at the driver’s seat and the roll velocity excited at the left and right tracks are compared in Figure 8, in which we can see the effect of the delay between the front- and rear-wheel excitations. From these comparisons, we see that the optimal active suspension with static output feedback achieves a weighted H2 norm that is approximately 10% smaller than that achieved by the optimal passive suspension.

5. CONCLUSION

In this paper, we show that design of a passive vehicle suspension on a multi-wheel model is equivalent to design of a structured controller and that H2 optimization is a meaningful approach to design optimization. The road roughness is taken as the disturbance input and various performance requirements are included in the cost
output. A Lagrange matrix multiplier is used to obtain a set of necessary conditions for optimization, taking into account auxiliary constraints such as vehicle symmetry. Based on this formulation, a time-efficient gradient-based approach is developed to solve directly for the H2 optimal design, minimizing the RMS value of the cost output.

As an example, we design passive suspensions for a four-wheel vehicle over a range of vehicle speeds and compare the performance of an adaptive suspension (in which parameters are adjusted according to vehicle speed) to that of a fixed-parameter suspension. We then apply the method developed in this paper to the design of an active suspension restricted to static output feedback and find that, for this example, an active static output feedback controller yields a 10% improvement in the weighted performance.

The framework provided in this paper can also be used for efficient design of passive as well as structure-constrained active engine mounts and other isolation systems.

REFERENCES


